# Pattern Formation in Competition-Diffusion equation

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**Abstract**

Tiger stripes, leopard spots, angelfish stripes, patterns on a zebra, patterns in sunflower (Helianthus) etc. are common patterns in real life. According to Alan Turing in 1952 [14], these patterns can be explained by some reaction-diffusion equation corresponding to the system. Conditions for patterns in the competition-diffusion equation in 1D are obtained in this paper.

**Keywords**

Diffusion equation, Reaction-Diffusion Equation, Pattern formation, Routh Hurwitz criteria

## 1 Introduction

Alan Turing in 1952 proposed that patterns observed in nature came as a result of interaction between two chemicals (which he called morphogens) which diffuse at different rates. One acts as the activator while the other acts as the inhibitor. This situation is possible if the system is stable in the absence of diffusion but unstable in the presence of diffusion. Several works on this subject include [1, 3, 4, 5, 7, 8, 10, 11, 12].
The reaction diffusion equation in 1D is of the form

$$\frac{\partial u}{\partial t} = f(u,v) + D_1\frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

$$\frac{\partial v}{\partial t} = g(u,v) + D_2\frac{\partial^2 v}{\partial x^2} \tag{1.2}$$

where $f(u,v)$ and $g(u,v)$ are the reaction parts, $D_1\frac{\partial^2 u}{\partial x^2}$ and $D_1\frac{\partial^2 u}{\partial x^2}$ are the diffusion parts and $D_1$ and $D_2$ are the diffusion coefficients. Equations (1.1) and (1.2) can be written in the matrix form as

$$w_t = F(w) + Dw_{xx}, \tag{1.3}$$

with

$$w = \begin{pmatrix} u \\ v \end{pmatrix},\ F(w) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix},\ D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \text{ and } w_{xx} = \frac{\partial^2}{\partial x^2}w. \tag{1.4}$$

Turing mechanism has been justified for population dynamics and several authours have worked on it [2, 6, 9, 15]. The application of the Turing mechanism to competition model is considered in this paper. The competition model is given as

$$\dot{x} = a_1 x \left(1 - \frac{x + \alpha_{12} y}{k_1}\right), \tag{1.5}$$

$$\dot{y} = a_2 y \left(1 - \frac{y + \alpha_{21} x}{k_2}\right), \tag{1.6}$$

where $\cdot x = \frac{dx}{d\tau}$, $\dot{y} = \frac{dy}{d\tau}$. By introducing the non-dimensional quantities

$$t = a_1 \tau,\ x = k_1 u,\ y = k_2 v,\ \alpha = \frac{a_2}{a_1},\ \beta_1 = \frac{\alpha_{12} k_2}{k_1},\ \beta_2 = \frac{\alpha_{21} k_1}{k_2},$$

the 6-parameter system (1.5) and (1.6) becomes a 3-parameter non-dimensionalised system

$$\frac{du}{dt} = u\left(1 - u - \beta_1 v\right), \tag{1.7}$$

$$\frac{dv}{dt} = \alpha v\left(1 - v - \beta_2 u\right). \tag{1.8}$$

Suppose $(u_0, v_0)$ is a steady state of the non-dimensionalised competition system of equations and putting

$$f(u,v) = u(1 - u - \beta_1 v), \quad \text{and} \quad g(u,v) = \alpha v(1 - v - \beta_2 u), \tag{1.9}$$

and

$$f_u = \left.\frac{\partial f}{\partial u}\right|_{(u_0,v_0)},\ f_v = \left.\frac{\partial f}{\partial v}\right|_{(u_0,v_0)},\ g_u = \left.\frac{\partial g}{\partial u}\right|_{(u_0,v_0)},\ g_v = \left.\frac{\partial g}{\partial v}\right|_{(u_0,v_0)},$$

the system is linearised to become

$$\frac{du}{dt} = (u - u_0) f_u + (v - v_0) f_v \tag{1.10}$$

$$\frac{dv}{dt} = (u - u_0) g_u + (v - v_0) g_v \tag{1.11}$$

which can be written in matrix form as

$$w_t = Aw$$

and the characteristic equation is the quadratic equation obtained from

$$|A - \lambda I| = 0.$$

Using the Routh Hurwitz criteria [13], we therefore require for stability in the absence of diffusion that

$$Tr(A) = f_u + g_v < 0 \tag{1.12}$$

$$det(A) = f_u g_v - f_v g_u > 0. \tag{1.13}$$

## 2 Diffusion-driven instability

We attempt to analyze the patterns formed from the competition-diffusion at the steady states. To start with, we seek a solution of the system 1.3 to be of the form

$$w = \sum_k c_k e^{\lambda t} W_k \quad \text{where} \quad W_k = \begin{pmatrix} \cos kx \\ \cos kx \end{pmatrix} \implies w_{xx} = -k^2 w, \quad \text{and } w_t = \lambda w \tag{2.1}$$

(and $k = \frac{n\pi}{r}$ is the wavenumber on the interval $[0,r]$) and we linearize $F(w)$ to get

$$F(w) = Aw.$$

With these, (1.3) becomes,

$$\lambda w = Aw - k^2 Dw \implies \left(A - k^2 D - \lambda I\right) w = 0. \quad (2.2)$$

Since $w \neq 0$, then the whole problem simply reduces to the eigenvalue problem

$$\left| \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} - k^2 \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0 \quad (2.3)$$

Let

$$J = \begin{pmatrix} f_u - k^2 D_1 & f_v \\ g_u & g_v - k^2 D_2 \end{pmatrix} \quad (2.4)$$

then the characteristic equation is

$$\lambda^2 - Tr(J)\lambda + det(J) = 0 \quad (2.5)$$

where

$$Tr(J) = f_u + g_v - k^2 (D_1 + D_2) \quad (2.6)$$

and

$$\begin{aligned} det(J) &= \left(f_u - k^2 D_1\right)\left(g_v - k^2 D_2\right) - f_v g_u, \\ &= D_1 D_2 k^4 - k^2 \left(f_u D_2 + g_v D_1\right) + f_u g_v - f_v g_u. \end{aligned} \quad (2.7)$$

Since we want instability in the presence of diffusion, then by Routh Hurwitz criteria, we require that

$$Tr(J) = f_u + g_v - k^2 (D_1 + D_2) > 0 \quad (2.8)$$

or

$$det(J) = D_1 D_2 k^4 - k^2 (f_u D_2 + g_v D_1) + f_u g_v - f_v g_u < 0. \quad (2.9)$$

We require that $f_u + g_v < 0$ from (1.12), and since $-k^2 (D_1 + D_2) < 0$, then $Tr(J) = f_u + g_v - k^2 (D_1 + D_2) < 0$ and consequently, condition (2.8) cannot hold. We are therefore left with
(2.9). Rewrite (2.9) as

$$h(m) = D_1 D_2 m^2 - (f_u D_2 + g_v D_1) m + f_u g_v - f_v g_u, \ (m = k^2). \quad (2.10)$$

The minimum of $h(m)$ occurs at the critical wavenumber $k_c$

$$m = k_c^2 = \frac{f_u D_2 + g_v D_1}{2 D_1 D_2} > 0 \quad (2.11)$$

$$\Rightarrow f_u D_2 + g_v D_1 > 0 \quad (2.12)$$

and we therefore require that

$$h_{min} = -\frac{(f_u D_2 + g_v D_1)^2}{4 D_1 D_2} + f_u g_v - f_v g_u < 0 \quad (2.13)$$

$$f_u g_v - f_v g_u < \frac{(f_u D_2 + g_v D_1)^2}{4 D_1 D_2} \quad (2.14)$$

The conditions, therefore, for diffusion-driven instability are

$$f_u + g_v < 0, \text{ from (1.12)} \tag{2.15}$$
$$f_u g_v - f_v g_u > 0, \text{ from (1.13)} \tag{2.16}$$
$$f_u D_2 + g_v D_1 > 0, \text{ from (2.12)} \tag{2.17}$$
$$f_u g_v - f_v g_u < \frac{(f_u D_2 + g_v D_1)^2}{4D_1 D_2}. \text{ from (2.14)} \tag{2.18}$$

Solving (2.10), we find that the unstable wavenumbers must fall between

$$k_1^2 = \frac{1}{2D_1D_2}\left(f_uD_2 + g_vD_1 - \sqrt{(f_uD_2+g_vD_1)^2 - 4D_1D_2(f_ug_v - f_vg_u)}\right) < k^2 < \frac{1}{2D_1D_2}\left(f_uD_2 + g_vD_1 + \sqrt{(f_uD_2+g_vD_1)^2 - 4D_1D_2(f_ug_v - f_vg_u)}\right) = k_2^2 \tag{2.19}$$

and

$$w = \sum_{k=n_1}^{n_2} c_k e^{\lambda t} W_k \text{ where} W_k = \begin{pmatrix}\cos kx\\ \cos kx\end{pmatrix} \tag{2.20}$$

where $n_1$ is the least integer greater than $k_1$ and $n_2$ is the greatest integer smaller than $k_2$.

# 3 Analysis of the competition-diffusion equation

We obtain the competition-diffusion equation by putting the competition equation as the reaction part of the reaction-diffusion equation. The model, therefore, is

$$\frac{\partial u}{\partial t} = u(1 - u - \beta_1) + D_1\frac{\partial^2 u}{\partial x^2} \tag{3.1}$$

$$\frac{\partial v}{\partial t} = \alpha v(1 - v - \beta_2 u) + D_2\frac{\partial^2 v}{\partial x^2} \tag{3.2}$$

It is easy to see that there are two steady states for the competition model which are (0,0) and $\left(\frac{1-\beta_1}{1-\beta_1\beta_2}, \frac{1-\beta_2}{1-\beta_1\beta_2}\right)$. This model is useful in studying the pattern that occurs in a market where there are competition between two different products.

## 3.1 Impossible pattern at the origin

The origin corresponds to the state where none of the products is available in the market. If we consider the steady state at the origin (0,0), we have

$$f_u = 1, f_v = 0, g_u = 0, g_v = \alpha. \tag{3.3}$$

Putting (3.3) into equations (2.15) through (2.18), we require that

$$1 + \alpha < 0, \qquad \alpha > 0 \tag{3.4}$$

$$D_2 + \alpha D_1 > 0, \qquad \alpha < \frac{(D_2 + \alpha D_1)^2}{4D_1 D_2} \tag{3.5}$$

Clearly, equation (3.4) cannot hold at the same time. Therefore, if the system starts from the origin (or somewhere close to the origin), no pattern is formed. This is the situation experienced when two competing products are not available in the market or only few pieces of each product are available in the market. There is no pattern to expect since there are no product to sell or all the few products (from the two products) get sold. This is just a trivial case.

## 3.2 Pattern at $\left(\frac{1-\beta_1}{1-\beta_1\beta_2}, \frac{1-\beta_2}{1-\beta_1\beta_2}\right)$

At this steady state, we have

$$u_0 = \frac{1-\beta_1}{1-\beta_1\beta_2},\ v_0 = \frac{1-\beta_2}{1-\beta_1\beta_2}$$

Substituting $f_u f_v, g_u, g_v$ into (2.15) through (2.18), we require that

$$f_u + g_v = -u_0 - \alpha v_0 < 0 \Rightarrow u_0 + \alpha v_0 > 0 \tag{3.6}$$

$$f_u g_v - f_v g_u = \alpha u_0 v_0 (1 - \beta_1\beta_2) > 0 \tag{3.7}$$

$$f_u D_2 + g_v D_1 = u_0 D_2 + \alpha v_0 D_1 < 0 \Rightarrow u_0 + \alpha v_0 D < 0 \tag{3.8}$$

and

$$f_u g_v - f_v g_u < \frac{(f_u D_2 + g_v D_1)^2}{4D_1 D_2}. \tag{3.9}$$

Equation (3.9) implies

$$\alpha u_0 v_0 (1 - \beta_1\beta_2) < \frac{(u_0 D_2 + \alpha v_0 D_1)^2}{4D_1 D_2} \tag{3.10}$$

$$D^2 - \frac{2u_0}{\alpha v_0}(1 - 2\beta_1\beta_2)D + \left(\frac{u_0}{\alpha v_0}\right)^2 > 0 \tag{3.11}$$

*Remark* 3.1 $D = \frac{D_1}{D_2} \neq 1$, because otherwise inequality (3.8) becomes

$$u_0 + \alpha v_0 < 0 \tag{3.12}$$

which contradicts (3.6).
The choice of the parameters $\beta_1$ and $\beta_2$ that will satisfy the four conditions are

$$0 < \beta_1 < 1,\ \beta_2 > \frac{1}{\beta_1} \text{ or } 0 < \beta_2 < 1,\ \beta_1 > \frac{1}{\beta_2}$$

Consider when $\beta_1 = \frac{1}{2},\ \beta_2 = 3$ so that Condition 3.6 and 3.7 become

$$(-1) + 4\alpha > 0 \Rightarrow \alpha > \frac{1}{4} \text{ and } \alpha(-1)(4)\left(-\frac{1}{2}\right) > 0 \Rightarrow \alpha > 0 \tag{3.13}$$

so that

$$\alpha > \frac{1}{4} \tag{3.14}$$

Condition 3.8 becomes

$$(-1) + 4\alpha D < 0 \implies D < \frac{1}{4\alpha} \tag{3.15}$$

Condition 3.11 becomes

$$D^2 - \frac{1}{\alpha}D + \frac{1}{16\alpha^2} > 0$$

which implies that

$$0 < D < \frac{2-\sqrt{3}}{4\alpha} \text{ or } D > \frac{2+\sqrt{3}}{4\alpha} \qquad (3.16)$$

Combining 3.15 and 3.16, we have

$$0 < D < \frac{2-\sqrt{3}}{4\alpha}. \qquad (3.17)$$

Thus, we have the range of values of $\alpha$ and $D$ as

$$\alpha > \frac{1}{4} \text{ and } 0 < D < \frac{2-\sqrt{3}}{4\alpha} \approx \frac{0.06699}{\alpha}$$

and the unstable wavenumbers fall between

$$k_1^2 = \frac{1}{2D_1}\left(1 - 4\alpha D - \sqrt{16\alpha^2D^2 - 16\alpha D + 1}\right) < k^2$$

$$< \frac{1}{2D_1}\left(1 - 4\alpha D + \sqrt{16\alpha^2D^2 - 16\alpha D + 1}\right) = k_2^2.$$

# 4 Conclusion

There are two steady states of the competition model and they are (0,0) and $\left(\frac{1-\beta_1}{1-\beta_1\beta_2}, \frac{1-\beta_2}{1-\beta_1\beta_2}\right)$. Our analysis revealed that the system cannot exhibit any pattern if it starts from anywhere near the origin. But, for the second steady state, patterns will occur if

$$0 < \beta_1 < 1,\ \beta_2 > \frac{1}{\beta_1} \text{ or } 0 < \beta_2 < 1,\ \beta_1 > \frac{1}{\beta_2}$$

So, with specific values of $\beta_1 = \frac{1}{2}$ and $\beta_2 = 3$, we need

$$\alpha > \frac{1}{4} \text{ and } 0 < D < \frac{2-\sqrt{3}}{4\alpha} \approx \frac{0.06699}{\alpha}.$$

and figures (4.1) below show the feasible region. The shaded regions are the regions where any choice of $\beta_1, \beta_2, \alpha$ and $D$ will produce a pattern and the pattern produced when $\beta_1 = \frac{1}{2}, \beta_2 = 3$, and $\alpha = 1$ is shown in figure (4.2) below

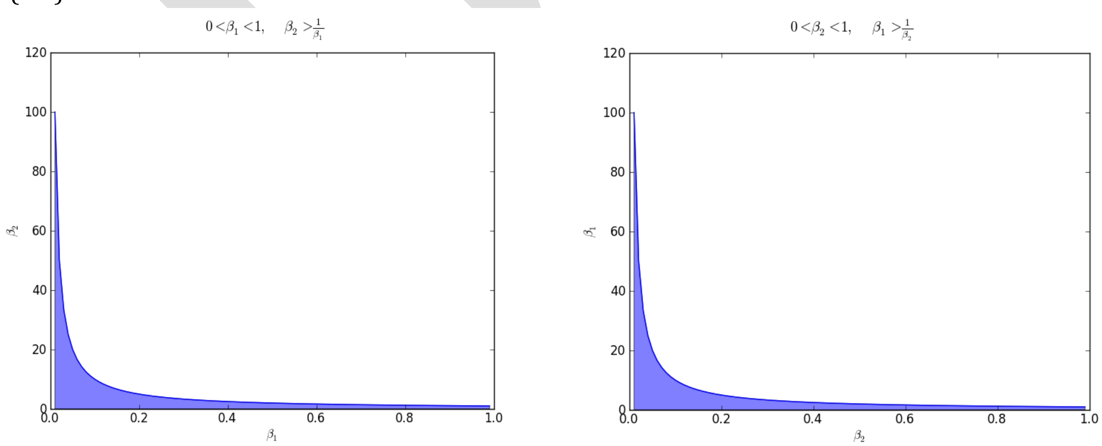


Figure 4.1: Feasible region

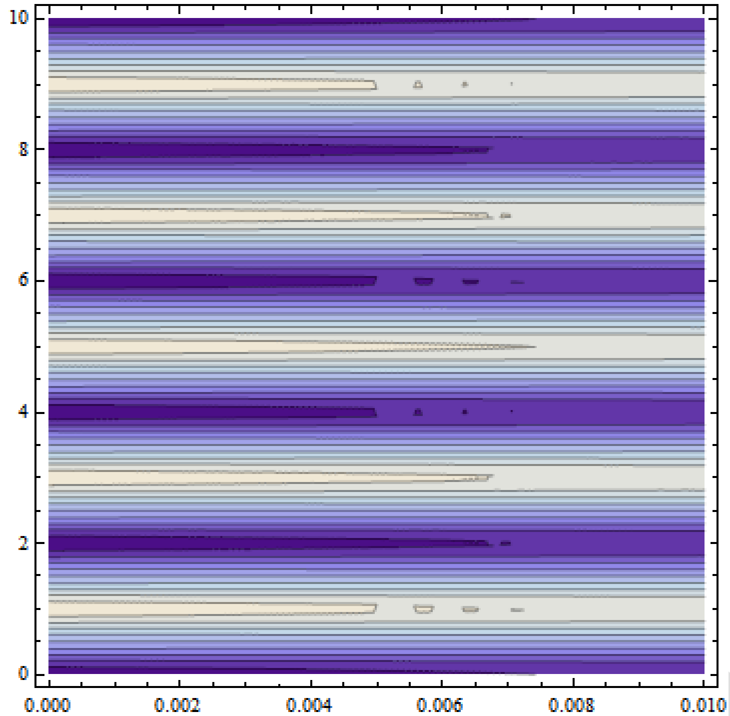


Figure 4.2: Pattern in the competition-diffusion equation with $\beta_1 = \frac{1}{2}, \beta_2 = 3$

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