

Research Article

On Performance of Two-Parameter Gompertz-Based \overline{X} Control Charts

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In this paper, two methods of control chart were proposed to monitor the process based on the two-parameter Gompertz distribution. The proposed methods are the Gompertz Shewhart approach and Gompertz skewness correction method. A simulation study was conducted to compare the performance of the proposed chart with that of the skewness correction approach for various sample sizes. Furthermore, real-life data on thickness of paint on refrigerators which are nonnormal data that have attributes of a Gompertz distribution were used to illustrate the proposed control chart. The coverage probability (CP), control limit interval (CLI), and average run length (ARL) were used to measure the performance of the two methods. It was found that the Gompertz exact method where the control limits are calculated through the percentiles of the underline distribution has the highest coverage probability, while the Gompertz Shewhart approach and Gompertz skewness correction method have the least CLI and ARL. Hence, the two-parameter Gompertz-based methods would detect out-of-control faster for Gompertz-based \overline{X} charts.

1. Introduction

Shewhart \overline{X} control chart is one of the most widely used statistical process control techniques developed to monitor the process average [1]. Shewhart control charts are often based on the assumption that the sample observations are independently and identically distributed and the process observations follow a normal distribution. However, skewed data often violate the normality assumption and cause an increase in error probability when control charts are used to monitor the process. Skew probability distributions always occur in the monitoring process of real-life data during the production process, but gamma distribution is often selected to examine the performances of control charts [2]. Furthermore, whenever skew distributions exist, mean and variance may not be appropriate summary statistics to measure the process variation [3]. When the distribution of data is known, the use of the exact method proposed to

provide accurate control limits is more likely to detect whether a process is in control or not [4, 5].

Some exact control limits based on the form of underlying distribution are investigated in the literature. Construction of control charts using the theory of confidence intervals, when the random variable follows inverse Gaussian distribution, is considered by Edgeman [6]. Edgeman [6] justified his assumption by relying on the central limit theorem for nonnormal processes whenever the sample size used for control charting is less than 10. However, quality characteristics are always better modelled by using probability distribution with nonnegative support rather than a normal distribution. Kantam and Sriram [7] developed control charts to be used when the process characteristic follows a gamma distribution. Kantam et al. [8] developed control charts for log-logistic distribution, Kan and Yazici [9] developed individual control charts for Burr distributed and Weibull distributed data, Subba Rao and Kantam [10] introduced control charts for double exponential distribution, Yazici and Kan [11] developed asymmetric control limits for small samples, Srinivasa Rao and Sarath Babu [12] proposed control charts for linear failure rate distribution, and Srinivasa Rao and Kantam [13] developed control charts for half-logistic distribution percentiles. Srinivasa Rao et al. [14] developed control charts for a variable quality characteristic that is assumed to follow size-biased Lomax distribution based on the evaluated percentiles of sample statistics like mean, median, midrange, range, and standard deviation, Srinivasa Rao et al. [15] developed control charts for a variable quality characteristic that is assumed to follow new Weibull-Pareto distribution based on the evaluated percentiles of sample statistics like mean, median, midrange, range, and standard deviation, Wang et al. [16] proposed control charts for monitoring the lower Weibull percentiles under complete data and Type-II censoring, and Rao [17] considered an exponentiated halflogistic distribution to develop an attribute control chart for time-truncated life tests with a known or unknown shape parameter and references therein.

In this paper, the \overline{X} Shewhart-type control chart based on Gompertz distribution is proposed for monitoring a nonnormal process. A simulation study was conducted to compare the performance of the proposed chart with that of the skewness correction approach for various sample sizes. Furthermore, real-life data on thickness of paint on refrigerators which are nonnormal data that have attributes of a Gompertz distribution were used to measure the performance of the proposed control chart.

2. Two-Parameter Gompertz Distribution

Gompertz distribution is an exponentially increasing, continuous probability distribution. It is basically a truncated extreme value distribution [18]. Gompertz distribution is a lifetime distribution and is often applied to describe the distribution of adult life spans by actuaries and demographers. It is considered for the analysis of survival in some sciences such as biology, gerontology, computer science, and marketing science [19].

The Gompertz distribution is an extension of the exponential distribution. According to Alizadeh et al. [20], the cumulative distribution function (CDF) and probability density function (PDF) of the Gompertz generalized family of distributions are given, respectively, by

$$F(x) = 1 - e^{(\theta/\gamma) \left\{ 1 - [1 - \overline{G}(x)]^{\gamma} \right\}}; \quad \theta > 0, \gamma > 0, \tag{1}$$

$$f(x) = \theta g(x) [1 - G(x)]^{-\gamma - 1} e^{(\theta/\gamma) \{1 - [1 - \overline{G}(x)]^{\gamma}\}}, \quad \theta > 0, \gamma > 0,$$
(2)

where θ and γ are additional shape parameters which are introduced to vary tail weights and G(x) and g(x) are, respectively, the CDF and PDF of the exponential distribution which is the parent distribution. The CDF and PDF of the exponential distribution with the parameter λ are given by the following equations:

$$G(x) = 1 - e^{-\lambda x}, \quad \lambda > 0, \tag{3}$$

$$g(x) = \lambda e^{\wedge} (-\lambda x), \quad \lambda > 0.$$
(4)

Therefore, the PDF of the Gompertz exponential distribution is derived by inserting the expression in equations (3) and (4) into equation (2). This gives

$$f(x) = \theta \lambda e^{\lambda x \gamma} e^{(\theta/\gamma) \left[1 - e^{\lambda x \gamma}\right]}, \quad x > 0, \ \theta > 0, \ \gamma > 0, \ \lambda > 0.$$
(5)

The CDF of the Gompertz exponential distribution is derived by inserting equation (3) into equation (1). Thus, we have the expression

$$F(x) = 1 - e^{(\theta/\gamma) \left[1 - e^{\lambda x \gamma}\right]}, \quad \theta > 0, \ \gamma > 0, \ \lambda > 0.$$
(6)

 γ , λ , and θ in (5) and (6) may be combined into two independent parameters, say, *t* and *z* defined as follows: $z = \theta/\gamma$ and $t = \lambda\gamma$.

Then, f(x) in equation (5) becomes

$$f(x) = tze^{tx}e^{z\left[1-e^{tx}\right]}, \quad x > 0, \ t > 0, \ z > 0.$$
(7)

The CDF, F(x) in equation (6), becomes

$$F(x) = 1 - e^{z \lfloor 1 - e^{tx} \rfloor}; \quad x > 0, \ t > 0, \ z > 0.$$
(8)

The expressions in equations (7) and (8) are, respectively, the PDF and CDF of Gompertz distribution with parameters *t* and *z*.

2.1. Gompertz \overline{X} Control Chart Using the Shewhart Approach. Consider the Shewhart \overline{X} chart which contains the center line (CL) that represents the average value of quality characteristics corresponding to the in-control state. There are two horizontal lines, namely, the lower control limit (LCL) and upper control limit (UCL). These control limits are selected so that if the process is in control, nearly all sample points will fall within them. If w is a statistic that measures the quality characteristic and if the mean and the variance of w are given as μ_{Gw} and σ_{Gw}^2 , respectively, then the general model for the Shewhart control chart is given as

upper control limit = UCL =
$$\mu_{Gw} + L\sigma_{Gw}$$
,
center line = CL = μ_{Gw} , (9)
lower control limit = LCL = $\mu_{Gw} - L\sigma_{Gw}$,

where *L* is the distance of the control limits from the center line.

In the construction of control charts, it is common to set L = 3. Vysochanskij and Petunin [21] refined Chebyshev's inequality by including the factor of 4/9 and made it possible to set 3-sigma limits for any unimodal distribution. The Vysochanskij–Petunin inequality allows the inference that, for any unimodal distribution, at least 95% of the data will be captured by limits placed at 3-sigma. Therefore, for Gompertz-based \overline{X} control charts, the control limits can be obtained as follows:

upper control limit = UCL_G =
$$\mu_G + 3\sqrt{\sigma_G^2}$$
,
center line = CL_G = μ_G , (10)
lower control limit = LCL_G = $\mu_G - 3\sqrt{\sigma_G^2}$.

When the process data are assumed to follow Gompertz distribution, we used the expression in equation (7) to derive the mean and variance of the Gompertz distribution (μ_G and σ_G^2). Using Maple software, the mean of Gompertz exponential distribution is derived in an integral form (see details in Appendix). The derived mean is given as

mean =
$$\mu_G = \frac{e^z}{t} \int_z^\infty \ln\left(\frac{w}{z}\right) e^{-w} dw = \frac{e^z Ei(1,z)}{t}.$$
 (11)

Similarly, the variance of the Gompertz exponential distribution is derived and presented in Appendix. The derived variance is

$$\sigma_G^2 = \frac{e^z}{t^2} \int_z^\infty \ln^2\left(\frac{w}{z}\right) e^{-w} dw - \left[\frac{e^z}{t} \int_z^\infty \ln\left(\frac{w}{z}\right) e^{-w} dw\right]^2$$

= $\frac{1}{6t^2} \left(e^z \left(-6e^z Ei(1,z)^2 + \pi^2 - 12 \text{ hypergeom}\left([1,1,1], [2,2,2], -z\right)z + 6 \ln(z)^2 + 12 \ln(z)\gamma + 6\gamma^2\right)\right),$ (12)

where Ei(a, z) is an exponential integral and hypergeom (.) is a generalized hypergeometric function.

2.2. Gompertz \overline{X} Control Chart Using the Skewness Correction Approach. The skewness correction (SC) method is used for constructing the \overline{X} control charts for skewed distributions. Its asymmetric control limits are obtained by taking into consideration the degree of skewness estimated from subgroups, and with no assumptions on the distributions. According to Chan and Cui [4], if the parameters of a process are known, the control limits of the \overline{X} control chart are given by

$$UCL_{\overline{X}} = \mu_{\overline{X}} + \frac{(3 + C_4^*)\sigma_{\overline{X}}}{\sqrt{n}},$$

$$LCL_{\overline{X}} = \mu_{\overline{X}} + \frac{(-3 + C_4^*)\sigma_{\overline{X}}}{\sqrt{n}},$$
(13)

where C_4^* is the control chart constant for the SC method. It should be noted that $C_4^* = 0$ if the underlying distribution is symmetric and the control limits of the SC method for \overline{X} control chart will reduce to the traditional control limits of the Shewhart \overline{X} control chart. However, if the distribution is asymmetrical, Chan and Cui [4] gave the constant C_4^* in equation (13) to be

$$C_4^* = \frac{(4/3)k_3(\overline{X})}{1 + 0.2k_3^2(\overline{X})},\tag{14}$$

where $k_3(\overline{X})$ is the skewness of the subgroup mean \overline{X} .

Let x_1, x_2, \ldots, x_n be samples from Gompertz distribution with mean μ_{Gw} and standard deviation σ_{Gw} . The control limits and the center line for a skewness correction method for the \overline{X} chart are

$$UCL_{\overline{X}} = \mu_{Gw} + \frac{(3 + C_4^*)\sigma_{Gw}}{\sqrt{n}},$$

$$CL_{\overline{X}} = \mu_{Gw},$$

$$LCL_{\overline{X}} = \mu_{Gw} + \frac{(-3 + C_4^*)\sigma_{Gw}}{\sqrt{n}}.$$
(15)

To obtain the constant C_4^* , Bowley's formula is chosen for finding the coefficient of skewness which is given by

$$k_{3G}(\overline{X}) = \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1},$$
(16)

where Q_i (*i* = 1, 2, 3) is the *i*th quartile of the Gompertz distribution.

Hence, the constant

$$C_4^* = \frac{(4/3)k_{3G}(\overline{X})}{1 + 0.2k_{3G}^2(\overline{X})},\tag{17}$$

where $k_{3G}(\overline{X})$ is the skewness of the subgroup mean \overline{X} .

2.3. Using Exact Control Limits. Let x_1, x_2, \ldots, x_n be a random sample subgroup of industrial process data of size n supposed to have been drawn from the two-parameter Gompertz exponential distribution with a targeted population average; under repeated sampling, the statistic \overline{X} gives whether the process average is around the targeted mean or not. Statistically speaking, we have to find the "most probable" limits within which \overline{X} falls. It is well known that 3σ limits of normal distribution include 99.73% of probability. Hence, we have to search for two limits of the sampling distribution such that the probability content of those limits is 0.9973 (see Srinivasa Rao et al. [14] and Srinivasa et al. [15]).

Symbolically, we have to find *L* and *U* such that

$$P(L \le \overline{x}_i \le U) = 0.9973, \tag{18}$$

where \overline{x}_i is the mean of sample size *n*. Taking the equitailed concept, *L* and *U* are, respectively, 0.00135 and 0.99865 percentiles of the sampling distribution of \overline{x}_i .

$$P\left(Z_{0.00135} \le \overline{X} \le Z_{0.99865}\right) = 0.9973,$$

$$P\left(\frac{Z_{0.00135}}{\mu_G}, \overline{x} \le \overline{x}_i \le \frac{Z_{0.99865}}{\mu_G}, \overline{x}\right) = 0.9973, \quad (19)$$

$$P\left(A_{2p}^*, \overline{x} \le \overline{x}_i \le A_{2p}^{**}, \overline{x}\right) = 0.9973,$$

where \overline{x} is the grand mean and \overline{x}_i is the *i*th subgroup mean. Thus, A_{2p}^* and A_{2p}^{**} are the percentile constants of the \overline{x} chart. Hence, the control limits and the center line for an exact \overline{X} control chart are

Sample size	Skewness correction		Gompertz Shewhart		Gompertz skewness correction		Gompertz exact		
	LCL	UCL	LCL	UCL	LCL	UCL	LCL	UCL	
2	0	0.617	0	0.2266	0	0.2287	0.0003	0.2810	
3	0	0.573	0.0135	0.1470	0.0126	0.1462	0.0005	0.1791	
4	0	0.491	0.0209	0.1515	0.0204	0.1510	0.0004	0.2004	
5	0	0.469	0.0315	0.1256	0.0309	0.1250	0.0006	0.1658	
7	0	0.402	0.0454	0.1169	0.0448	0.1163	0.0009	0.1555	
10	0	0.328	0.0563	0.1018	0.0560	0.1014	0.0021	0.1303	

TABLE 1: Control limits for skewness correction, Gompertz Shewhart, Gompertz skewness correction, and Gompertz exact control charts.

TABLE 2: Coverage probabilities.

Sample size	Skewness correction	Gompertz Shewhart	Gompertz skewness correction	n Gompertz exact	
2	1.0	0.989	0.989	0.998	
3	1.0	0.918	0.917	0.978	
4	1.0	0.947	0.947	0.997	
5	1.0	0.875	0.875	0.988	
7	1.0	0.825	0.826	0.991	
10	1.0	0.695	0.695	0.976	

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Subgroup	1	2	3	4	5
1	2.7	2.3	2.6	2.4	2.7
2	2.6	2.4	2.6	2.3	2.8
3	2.3	2.3	2.4	2.5	2.4
4	2.8	2.3	2.4	2.6	2.7
5	2.6	2.5	2.6	2.1	2.8
6	2.2	2.3	2.7	2.2	2.6
7	2.2	2.6	2.4	2.0	2.3
8	2.8	2.6	2.6	2.7	2.5
9	2.4	2.8	2.4	2.2	2.3
10	2.6	2.3	2.0	2.5	2.4
11	3.1	3.0	3.5	2.8	3.0
12	2.4	2.8	2.2	2.9	2.5
13	2.1	3.2	2.5	2.6	2.8
14	2.2	2.8	2.1	2.2	2.4
15	2.4	3.0	2.5	2.5	2.0
16	3.1	2.6	2.6	2.8	2.1
17	2.9	2.4	2.9	1.3	1.8
18	1.9	1.6	2.6	3.3	3.3
19	2.3	2.6	2.7	2.8	3.2
20	1.8	2.8	2.3	2.0	2.9
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TABLE 3: Thickness of paint on refrigerators.

UCL =
$$A_{2p}^{**}.\overline{x}$$
,
CL = $\overline{\overline{x}}$, (20)
LCL = $A_{2p}^{*}.\overline{\overline{x}}$.



2.4. Performance Evaluation. In this work, three indicators were used to measure the performance of the derived control charts discussed above. They are the coverage probability (CP), control limit interval (CLI), and average run length (ARL). The coverage probability is the

FIGURE 1: Density and Q-Q plots of the thickness of paint data.

probability of the number of points within the control limits. It is used to compare the stability of the simulated process under different methods. The CLI and ARL were



FIGURE 2: CDF of the data and hypothetical Gompertz distribution.



FIGURE 3: Skewness correction \overline{X} control chart.



FIGURE 4: Gompertz Shewhart \overline{X} control chart.



FIGURE 5: Gompertz skewness correction \overline{X} control chart.



FIGURE 6: Gompertz exact \overline{X} control chart.

TABLE 4: Performance index values.

Skewness correction			Gompertz Shewhart			Gompertz skewness correction			Gompertz exact		
СР	CLI	ARL	СР	CLI	ARL	СР	CLI	ARL	СР	CLI	ARL
0.05	0.0472	1.0526	0.95	0.802	20	0.95	0.802	20	1.0	2.006	∞

used to compare the performance of the different methods for the real-life data.

3. Simulation Study

The steps for the simulation study are presented as follows.

Step 1: Control Chart Construction

- (1) Generate *n* independent Gompertz (2.2, 4.7) variates for *n* = 2, 3, 4, 5, 7, 10
- (2) Repeat Step (1) 30 times (r=30)
- (3) Compute the control limits for the X chart using equation (10) for the Shewhart approach, equation (15) for the skewness correction approach, and equation (20) for the exact approach

Step 2: Control Chart Operation

- For sample size n = 2, 3, 4, 5, 7, 10, generate a random subgroup from Gompertz (2.2, 4.7) variates
- (2) Repeat Step (1) 100 times (r = 100)

- (3) Compute the sample mean \overline{X}
- (4) Record whether the sample mean \overline{X} is out of the control limits of Step 3 and estimate a coverage probability for all methods
- (5) Repeat Steps 1 through 4 and obtain an average coverage probability for each control chart

4. Results

Using the two steps presented in Section 3, data were generated from a class of distribution of Gompertz with parameters t = 2.2 and z = 4.7 using Monte Carlo simulation. The coefficient of skewness of the generated data is 0.9226. The control limits for the two proposed methods and the exact method were computed to determine the stability of the simulated process. The coverage probability of the \overline{X} charts based on classical Shewhart was adopted. The results obtained were compared with those of the skewness correction method. The control limits of the skewness correction method and the Gompertz Shewhart approach, Gompertz skewness correction method, and Gompertz exact method using equations (10), (15), and (20), respectively, are computed and presented in Table 1.

To measure the performance of the charts, the coverage probability was computed for the four methods. The obtained results are presented in Table 2.

4.1. Data on Thickness of Paint on Refrigerators. The data on the thickness of paint on refrigerators presented in Table 3 were obtained from Priya and Kantam [22]. The data are for 20 subgroups of size n = 5 from a process that is known to be in control.

To test for the normality of the data in Table 3, the density and the Q-Q plots of the mean of the data are determined and presented in Figure 1. Furthermore, the Jarque–Bera normality test was obtained to be 10.556 with a p value of 0.005.

From Figure 1, it is clear that the data are not normally distributed. The p value of the Jarque–Bera normality also confirmed the nonnormality of the data. Furthermore, the skewness coefficient was computed to be 1.2195. Hence, the data are skewed data. Based on the fact that the data are skewed data, the Gompertz distribution was used to model the data, and it was discovered that the CDF of the data is approximately Gompertz (see Figure 2). Thus, the data are approximately a Gompertz random variable. Hence, we can use the Gompertz-based \overline{X} charts to monitor the process that produces the data.

The mean of the thickness of paint of one hundred refrigerators in twenty subgroups for sample size 5 was computed and used to compute the control limits derived in Section 2. The skewness correction, Gompertz Shewhart, Gompertz skewness correction, and Gompertz exact methods are used for the construction of the \overline{X} charts. The control charts for the four methods are presented in Figures 3–6.

The performance indices, namely, coverage probability (CP), control limit interval (CLI), and average run length (ARL), for the four methods are presented in Table 4.

5. Discussion of Results

The results of the control limits in Table 1 for the proposed methods showed improved limits compared to the skewness correction limits. The results of the coverage probabilities in Table 2 revealed that as the sample size increases, the coverage probabilities decrease for Gompertz Shewhart and Gompertz skewness correction methods. However, the coverage probabilities for skewness correction and Gompertz exact methods are more stable than those of Gompertz Shewhart and Gompertz skewness correction methods. The obtained results of the real-life data in Table 4 showed that the CLI and ARL of Gompertz Shewhart and Gompertz skewness correction methods are smaller but those of the Gompertz exact method are the smallest.

6. Conclusion

This study proposed control limits for the Gompertz \overline{X} control chart using different approaches. Coverage probability was used to compare the stability of the methods. The

results show that as the sample size increases, the coverage probabilities for the Gompertz Shewhart approach and Gompertz skewness correction approach decrease. The coverage probabilities for skewness correction and Gompertz exact methods are more stable than those of the Gompertz Shewhart approach and Gompertz skewness correction method. The results of the real-life data using the CLI and ARL showed that the Gompertz Shewhart approach and Gompertz skewness correction method would be able to detect out-of-control faster than the exact method for Gompertz-based \overline{X} charts. The CLI and ARL of the skewness correction method are too small and thereby would raise false alarm when there are none. Hence, it is recommended that the Gompertz Shewhart approach and Gompertz skewness correction method can be used to monitor skewed process data that have the attribute of a two-parameter Gompertz distribution.

Appendix

A: Derivation of Mean and Variance of the Gompertz Distribution $(\mu_G \text{ and } \sigma_G^2)$

When the process data are assumed to follow Gompertz distribution, we used the expression in equation (7) to derive the mean and variance of the Gompertz distribution (μ_G and σ_G^2). Using Maple software, the mean and variance of the Gompertz distributions are obtained as follows.

$$E(x) = \int_0^\infty x f(x) \mathrm{d}x. \tag{A.1}$$

Using (5) for f(x), we obtain

$$E(X) = \int_{0}^{\infty} x \theta \lambda e^{\lambda x \gamma} e^{(\theta/\gamma) \left[1 - e^{\lambda x \gamma}\right]} dx.$$
(A.2)

Let $t = \lambda \gamma$, $p = \theta \lambda$, and $z = \theta / \gamma$, then

$$E(X) = \int_{0}^{\infty} x p e^{z} e^{tx} e^{-z e^{tx}} dx.$$
 (A.3)

Let $s = pe^z$, then we have

$$E[X] = \int_0^\infty x s e^{tx} e^{-z e^{tx}} \mathrm{d}x.$$
(A.4)

Let $e^{tx} = u/z$, therefore, $du = zte^{tx}dx$ and $dx = (du/zte^{tx}) = (du/tu)$.

When x = 0, $u = ze^{tx} = ze^{t(0)} = ze^{0} = z$. When $x = \infty$, $u = ze^{tx} = ze^{t(\infty)} = ze^{\infty} = \infty$. From equation (7),

$$E(X) = \int_{0}^{\infty} xse^{tx}e^{-ze^{tx}}dx$$

= $\int_{z}^{\infty} xs\frac{u}{z}e^{-u}\frac{du}{tu}$ (A.5)
= $\int_{z}^{\infty} \frac{s}{zt}xe^{-u}du = \frac{s}{zt}\int_{z}^{\infty} xe^{-u}du.$

But
$$u = ze^{tx}$$
, $e^{tx} = u/z$, $tx = \ln(u/z)$, and $x = (1/t)\ln(u/z)$.
Recall that $(X) = (s/zt) \int_{z}^{\infty} xe^{-u} du$, therefore,
 $E(X) = \frac{s}{zt} \int_{z}^{\infty} \frac{1}{t} \ln \frac{u}{z} e^{-u} du$
 $= \frac{s}{zt^2} \int_{z}^{\infty} \ln \frac{u}{z} e^{-u} du$
 $= \frac{pe^z}{(\theta/\gamma)\lambda^2 \gamma^2} \int_{z}^{\infty} \ln \frac{u}{z} e^{-u} du$ (A.6)
 $= \frac{\theta\lambda e^{\theta/\gamma}}{(\theta/\gamma)\lambda^2 \gamma^2} \int_{z}^{\infty} \ln \frac{u}{z} e^{-u} du$
 $= \frac{e^{(\theta/\gamma)}}{\lambda \gamma} \int_{z}^{\infty} \ln \frac{u}{z} e^{-u} du$.

The mean of Gompertz exponential distribution is hereby derived in an integral form as

$$E(X) = \frac{e^{(\theta/\gamma)}}{\lambda\gamma} \int_{z}^{\infty} \ln \frac{u}{z} e^{-u} du, \qquad (A.7)$$

where

$$z = \frac{\theta}{\gamma},$$

$$t = \lambda\gamma,$$
 (A.8)

$$u = w = ze^{tx} = \frac{\theta}{\gamma} e^{\lambda\gamma x}.$$

Then, the mean becomes

mean =
$$\mu_G = \frac{e^z}{t} \int_z^\infty \ln \frac{w}{z} e^{-w} dw = \frac{e^z Ei(1, z)}{t}.$$
 (A.9)

The variance of the Gompertz exponential distribution is derived as follows: If X is a Gompertz random variable, then the PDF of X is

$$f(x) = \theta \lambda e^{\lambda x \gamma} e^{(\theta/\gamma) \left[1 - e^{\lambda x \gamma}\right]}.$$
 (A.10)

Then, the variance of X is derived using the expression

$$\operatorname{Var}(X) = E(X^2) - [E(X)]^2,$$
 (A.11)

where

$$E(X^{2}) - \int_{0}^{\infty} x^{2} f(x) dx - \int_{0}^{\infty} x^{2} \theta \lambda e^{\lambda x \gamma} e^{(\theta/\gamma) \left[1 - e^{\lambda x \gamma}\right]} dx.$$
(A.12)

Let $t = \lambda \gamma$, $p = \theta \lambda$, and $z = \theta / \gamma$, then

$$E(X^{2}) = \int_{0}^{\infty} x^{2} p e^{tx} e^{z \left[1 - e^{tx}\right]} dx$$
$$= \int_{0}^{\infty} x^{2} p e^{tx} e^{z} e^{-z e^{tx}}$$
$$= \int_{0}^{\infty} x^{2} p e^{z} e^{tx} e^{-z e^{tx}} dx.$$
(A.13)

Let
$$s = pe^z$$
, then $E(X^2)$ becomes

$$E(X^{2}) = \int_{0}^{\infty} x^{2} s e^{tx} e^{-ze^{tx}} dx.$$
 (A.14)

Let $u = ze^{tx}$, therefore, $e^{tx} = u/z$, $du = zte^{tx}dx$, and $dx = (du/zte^{tx}) = (du/tu)$.

When x = 0, $u = ze^{tx} = ze^{t(0)} = ze^{0} = z$. When $x = \infty$, $u = ze^{tx} = ze^{t(\infty)} = ze^{\infty} = \infty$. Then,

$$E(X^{2}) = \int_{0}^{\infty} x^{2} s e^{tx} e^{-ze^{tx}} dx$$
$$= \int_{z}^{\infty} x^{2} s \frac{u}{z} e^{-u} \frac{du}{tu}$$
$$= \int_{z}^{\infty} \frac{s}{zt} x^{2} e^{-u} du$$
$$= \frac{s}{zt} \int_{z}^{\infty} x^{2} e^{-u} du.$$
(A.15)

But $e^{tx} = (u/z)$, $tx = \ln(u/z)$, $x = (1/t)\ln(u/z)$, and $x^2 = ((1/t)\ln(u/z))^2 = (1/t^2)(\ln(u/z))^2$. Hence,

$$E(X^{2}) = \frac{s}{zt^{3}} \int_{z}^{\infty} \left(\ln \frac{u}{z}\right)^{2} e^{-u} du$$

$$= \frac{pe^{z}}{(\theta/\gamma)\lambda^{3}\gamma^{3}} \int_{z}^{\infty} \left(\ln \frac{u}{z}\right)^{2} e^{-u} du$$

$$= \frac{\theta\lambda e^{\theta/\gamma}}{(\theta/\gamma)\lambda^{3}\gamma^{3}} \int_{z}^{\infty} \left(\ln \frac{u}{z}\right)^{2} e^{-u} du$$

$$= \frac{e^{\theta/\gamma}}{\lambda^{2}\gamma^{2}} \int_{z}^{\infty} \left(\ln \frac{u}{z}\right)^{2} e^{-u} du.$$
 (A.16)

Using the expression derived above, the variance of the Gompertz exponential distribution is

$$\operatorname{Var}(X) = \frac{e^{(\theta/\gamma)}}{\lambda^2 \gamma^2} \int_{z}^{\infty} \left(\ln \frac{u}{z}\right)^2 e^{-u} du$$

$$-\left(\left(\frac{e^{(\theta/\gamma)}}{\lambda\gamma} \int_{z}^{\infty} \left(\ln \frac{u}{z}\right)^2 e^{-u} du\right)^2\right),$$
(A.17)

where $z = (\theta/\gamma)$, $t = \lambda\gamma$, and $u = w = ze^{tx} = (\theta/\gamma)e^{\lambda\gamma x}$, and then the variance becomes

$$\sigma_G^2 = \frac{e^z}{t^2} \int_z^\infty \ln^2\left(\frac{w}{z}\right) e^{-w} dw - \left[\frac{e^z}{t} \int_z^\infty \ln\left(\frac{w}{z}\right) e^{-w} dw\right]^2$$

= $\frac{1}{6t^2} \left(e^z \left(-6e^z Ei(1,z)^2 + \pi^2 - 12 \text{ hypergeom}\left([1,1,1], [2,2,2], -z)z + 6\ln(z)^2 + 12\ln(z)\gamma + 6\gamma^2\right)\right),$
(A.18)

where Ei(a, z) is an exponential integral and hypergeom (·) is a generalized hypergeometric function.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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