Probability of getting to N point in a game of two players

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ABSTRACT

Suppose two Players A and B are playing a game. They are playing to n number of games (n is integer) and the current result is 2-1 in favour of A. Suppose they are to stop abruptly and want to find out who had the biggest chance of winning. Each of them has 50% chance of winning a game. We tackled the problem of how big is A's chance of winning in this paper and a Mathematical model was derived to take care of n number of points where n is an integer. The Mathematical model was used to calculate some probabilities of getting to a point first in a game involving two persons. The results were tabulated and a graph drawn for pictorial view of the results. We discovered that the probability of getting to a point first is tending to unity as the number of games progresses.

Game , Probability , **mathematical model** , **Series**.

INTRODUCTION

The study of probability of winning a game is of paramount importance to game theorists, since there is a belief that every transaction in life is game. In this paper we derived a model for probability of getting to a point first in a game of two players starting with an initial condition of 2:1 in favour of Player A or 1:2 in favour of player B. The model was derived using the summation of series method.

Probability is a measure or estimation of how likely it is that something will happen or that a statement is true Franklin (2001), Probabilities take values between 0 (0% chance or *will not happen*) and 1 (100% chance or *will happen*) inclusive. The higher the degree of probability, the more likely the occurrence of event, or, in a longer series of samples, the greater the number of times such event is expected to occur. Hájek (2012).

These concepts have been given an axiomatic mathematical derivation in probability theory , which is widely used in such areas as mathematics, statistics, finance, gambling, science, artificial intelligence/machine learning and philosophy to draw inferences about the expected frequency of events. Probability theory is also used to describe the underlying mechanics and regularities of complex systems Hájek(2012)

When dealing with experiments that are random and well-defined in a purely theoretical setting (like tossing a fair coin), probability function attaches equal probability valu to each outcome, (tossing a fair coin twice will yield head/head with probability 1/4, because the four outcomes head/head, head/tails, tails/head and tails/tails are equally likely to occur). When it comes to practical application, however, the word *probability* does not have a singular direct definition. In fact, there are two major categories of probability interpretations, whose adherents possess conflicting views about the fundamental nature of probability: Wikipedia (2013)

Objectivists assign numbers to describe some objective or physical state of affairs. The most popular version of objective probability is frequentist probability, which claims that the probability of a random event denotes the *relative frequency of occurrence* of an experiment's outcome, when an experiment is repeated. This interpretation considers probability to be the relative frequency of outcomes "in the long run". A modification of this is propensity probability, which interprets probability as the tendency of some experiment to yield a certain outcome, even when it is performed only once.

Subjectivists assign numbers per subjective probability, i.e., as a degree of belief. The degree of belief has been interpreted as, "the price at which you would buy or sell a bet that pays 1 unit of utility if E, 0 if not E." The most popular version of subjective probability is Bayesian probability, which involves expert knowledge as well as experimental data to produce probabilities. The expert knowledge is represented by some (subjective) prior probability distribution. The data is incorporated in a likelihood function. The product of the prior and the likelihood, normalized, resulting to a posterior probability distribution that incorporates all the information known to date. Starting from arbitrary, subjective probabilities for a group of agents, some Bayesians claim that all agents will eventually have sufficiently similar assessments of probabilities, given enough evidence.

Probability theory is the branch of mathematics concerned with the analysis of random phenomena. Encyclopaedia Britannica (2012) The central objects of probability theory are random variables, stochastic processes, and events: mathematical abstractions of non-deterministic events or measured quantities that may either be single occurrences or evolve over time in an apparently random fashion. If an individual coin toss or the roll of dice is considered to be a random event, then if repeated many times the sequence of random events will exhibit certain patterns, which can be studied and predicted. Two representative mathematical results describing such patterns are the law of large numbers and the central limit theorem.

As a mathematical foundation for statistics, probability theory is essential to many human activities that involve quantitative analysis of large sets of data. Methods of probability theory also apply to descriptions of complex systems given only partial knowledge of their state, as in statistical mechanics. A great discovery of twentieth century physics was the probabilistic nature of physical phenomena at atomic scales, described in quantum mechanics.

The mathematical theory of probability has its roots traced to the attempts to analyze games of chance by Gerolamo Cardano in the sixteenth century, and by Pierre de Fermat and Blaise Pascal in the seventeenth century (for example the "problem of points"). Christiaan Huygens published a book on the subject in 1657 and in the 19th century a big work was done by Laplace in what can be considered today as the classic interpretation (Wikipedia 2012)

Initially, probability theory mainly considered **discrete** events, and its methods were mainly combinatorial. Eventually, analytical considerations compelled the incorporation of **continuous** variables into the theory.

This culminated in modern probability theory, on foundations laid by Andrey Nikolaevich Kolmogorov. Kolmogorov combined the notion of sample space, introduced by Richard von Mises, and **measure theory** and presented his axiom system for probability theory in 1933. Which later became the mostly undisputed axiomatic basis for modern probability theory but alternatives exist. Glenn(2012)

RELATED WORKS

Paul K. Newton and Joseph B. Keller in their work ‘Probability of Winning at Tennis I. Theory and Data in (2005) computed probability of winning game, a set and a match in a tennis game based on each players probability of winning a point on serve which they assume are independent identically distributed (iid) random variables.

Andrew (2007) used a simple model of backgammon to approximate the chances each player has of winning, and a readily computable and sufficiently accurate approximation of that is developed. They compare the model to simulated backgammon games, and the previous approximation was modified to fit the real data.

In the paper, Backgammon, doubling the stakes, and Brownian motion by written by Jochen Blath and Peter Mörters(2001) some of the properties of the doubling cube were analysed

THE PROBLEM

Players A and B are playing a game. They are playing a game *n times* (n is integer) and the current result is 2-1 in favour of player A. Suppose they are to stop abruptly and were to find out the player who has the highest chance of winning subsequent games.

Each of the players has 50% chance of winning a game.

Starting with the score 2:1 in favour of A, and the next game is equally likely to go to either of them:

```
     +-- 3:1
     |
2:1 --+
     |
     +-- 2:2
```

We can keep building that tree out with two outcomes for the next game in each case. In cases where A's score reaches 4, we'll mark it with an A, and where B reaches 4 We'll use a B:

```
                +-- 4:1 A
                |
      +-- 3:1 --+
      |         |
      |         +-- 3:2
      |
2:1 --+
      |          +-- 3:2
      |          |
      +-- 2:2 --+
                 |
                 +-- 2:3
```

At this level, we have a win for A. The probability of that win is $(1/2)^2$, or 1/4. So after two games, the probability is 1/4 that A wins, and 3/4 that play continues.

By playing the third game:

```
                  +-- 4:1 A
                  |
      +-- 3:1 --+
      |         |          +-- 4:2 A
      |         |          |
      |         +-- 3:2 --+
      |                    |
      |                    +-- 3:3
      |
2:1 --+
      |                    +-- 4:2 A
      |                    |
      |          +-- 3:2 --+
      |          |         |
      |          |         +-- 3:3
      |          |
      +-- 2:2 --+
                 |
                 |          +-- 3:3
                 |          |
                 +-- 2:3 --+
                            |
                            +-- 2:4 B
```

Looking at the second 4:2, for instance. Half the time, B will win the first game; after that, half the time A will win the second game; and half the time, A will win the third game. So the probability of ending up here is $(1/2)^3 = 1/8$.

Similar reasoning for the other cases gives us,

```
                +-- 4:1 A (1/2)^2
                |
      +-- 3:1 --+
      |         |         +-- 4:2 A (1/2)^3
      |         |         |
      |         +-- 3:2 --+
      |                   |
      |                   +-- 3:3 (1/2)^3
      |
2:1 --+
      |                   +-- 4:2 A (1/2)^3
      |                   |
      |         +-- 3:2 --+
      |         |         |
      |         |         +-- 3:3 (1/2)^3
      |         |
      +-- 2:2 --+
                |
                |         +-- 3:3 (1/2)^3
                |         |
                +-- 2:3 --+
                          |
                          +-- 2:4 B (1/2)^3
```

As a quick check the combined probability of a win (for either player) at this point is

$$\frac{1}{4} + \frac{3}{8} = \frac{5}{8}$$

And the combined probability of having reached 3:3 without a winner is

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Note that the probabilities of having or not having a winner add to 1, as they should.

And if there's a 3:3 tie, that means the next game decides the contest. So:

```
                   +-- 4:1 A (1/2)^2
                   |
        +-- 3:1 --+
        |          |         +-- 4:2 A (1/2)^3
        |          |         |
        |          +-- 3:2 --+
        |                    |         +-- 4:3 A (1/2)^4
                             |         |
        |                    +-- 3:3 --+
        |                              |
        |                              +-- 3:4 B (1/2)^4
        |
  2:1 --+
        |                    +-- 4:2 A (1/2)^3
        |                    |
        |                    |
        |          +-- 3:2 --+         +-- 4:3 A (1/2)^4
        |          |         |         |
        |          |         +-- 3:3 --+
        |          |                   |
        |          |                   +-- 3:4 B (1/2)^4
        |          |
       +-- 2:2 --+
                   |                   +-- 4:3 A (1/2)^4
                   |                   |
                   |         +-- 3:3 --+
                   |         |         |
                   |         |         +-- 3:4 B (1/2)^4
                   +-- 2:3 --+
                             |
                             +-- 2:4 B (1/2)^3
```

And now we can add up the probability of A's wins,

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{11}{16}$$

and the probability of B's wins,

$$\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{5}{16}$$

and they add up to 1, which they should--since one of them HAS to win 4 games eventually. The natural question is that what will the probability looks like if both players are going to n instead 4?

By induction the solution above can be generalized to n points

Suppose we have from the solution

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}$$

Up to 4 points then, if we go on further to 5 point the series will be

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}$$

Also to 6 point we have

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{64}+\frac{1}{64}+\frac{1}{64}+\frac{1}{64}+\frac{1}{64}$$

To 7 point we have,

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{64}+\frac{1}{64}+\frac{1}{64}+\frac{1}{64}+\frac{1}{64}+\frac{1}{128}+\frac{1}{128}$$
$$+\frac{1}{128}+\frac{1}{128}+\frac{1}{128}+\frac{1}{128}$$

This can be arranged as,

$$\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^4}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^5}+\frac{1}{2^5}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^6}+\frac{1}{2^6}+\frac{1}{2^6}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^7}$$
$$+\frac{1}{2^7}+\frac{1}{2^7}+\frac{1}{2^7}+\frac{1}{2^7}$$

$= 1.\frac{1}{2^2}+2.\frac{1}{2^3}+3.\frac{1}{2^4}+4.\frac{1}{2^5}+5.\frac{1}{2^6}+6.\frac{1}{2^7}$

So to r points we should have by induction,

$$1.\frac{1}{2^2}+2.\frac{1}{2^3}+3.\frac{1}{2^4}+4.\frac{1}{2^5}+5.\frac{1}{2^6}+6.\frac{1}{2^7}+\cdots+(r-1).\frac{1}{2^r}$$

Which equal,

$$\sum_{r=2}^{n}\frac{r-1}{2^r}$$

So the probability of player A getting to n point before player B provided they started from 2:1 in favor of player A is:

$$P(A\ get\ to\ n\ before\ B)=\sum_{r=2}^{n}\frac{r-1}{2^r}$$

Since the probability must add up to 1 then the probability of player B to get to n point before player A is:

$P(B\ get\ to\ n\ before\ A)=1-\sum_{r=2}^{n}\frac{r-1}{2^r}$

Using MATLAB and the command r=[2: n]

$P=sum((r.*1-1)./(2.\wedge r.*1))$ we have table 1 below,

N UMBER OF POINTS	PROBABILITY
2	0.25
3	0.5
4	0.6875
5	0.8125
6	0.8906
7	0.9375
8	0.9648
9	0.9805
10	0.9893
11	0.9941
12	0.9968
13	0.9983
14	0.9991
15	0.9995
16	0.9997
17	0.9999
18	0.9999
19	1.0000
20	1.0000

Table 1

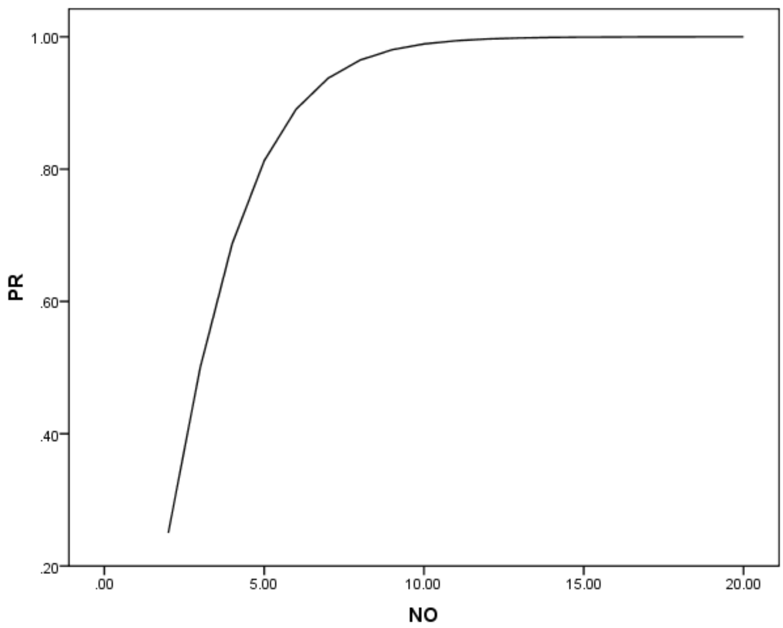

Figure 1

In figure 1 the probability P is rising as N progresses, but at a point n=19 the probability converges to unity, so the probability of getting to a point first will tend to 1 as the number of game progress

CONCLUSION

We derived a mathematical model for probability of getting to a point first in a game of two players starting with initial condition of 2:1 in favour of Player A or 1:2 in favour of player B. The model was derived using the series method.

After the derivation, we used the model to calculate Probabilities of getting to some point first by a player in a game of two persons.

The results were tabulated and a graph drawn for pictorial view of the results.

We discovered that the probability of getting to a point first is tending to unity as the number of games progresses.

RECOMMENDATIONS

For further investigation, the condition of starting at 2:1 in favour of A should be dropped and a new condition of 0:0 be adopted. Further investigation should be made on the behaviour of the new model for large values of N, the convergence of the model should be tested

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