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Solution to Nonlinear First and Second Order Differential Equations Using Mahgoub Transform Decomposition Method

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Abstract

A new scheme, Mahgoub transform Decomposition Method (MTDM) for the approximate solution of nonlinear first and second order differential equation with constant and variable coefficients is presented in this work. The new scheme is formulated using decomposition technique and the application of a recent Mahgoub transform method to obtain the analytic solution that approximate the exact exact solution of these equations. Existing examples are considered using the new scheme. The results show that the scheme is efficient, accurate and with Pade approximant application, the scheme converges faster with fewer terms in the iteration processes.

Keywords: Adomian Decomposition method, Adomian Polynomials, Mahgoub Transform method, Nonlinear Differential Equations, Pade approximant.

INTRODUCTION

This work considers a numerical scheme, Mahgoub Transform Decomposition Method(MTDM) numerical scheme, that can compliment analytic method using method of decomposition for solution of first and second order nonlinear differential equations with constant and variable coefficients. Mahgoub transform is more recent than Laplace transform that has been in application for some decades. The types of equations considered are second order initial value problems. We attempted to approximate the solution to these types of problems by transforming the equations and adopting the Adomian decomposition method to determine the nonlinear terms of the equation as introduced in (Adomian, 1991).The differential equation is transformed using Mahgoub Transform method. To solve the nonlinear part of the equation, an infinite series of the form $v=\sum_{i=0}^{\infty} v_i$ is assumed to the nonlinear terms, decomposed in terms of Adomian polynomials and then applied to the equation. The $v_i{}'s$ are determined then iteratively.

Nonlinear second order differential equation with initial value in form of

$$\frac{d^n y}{dx^n}+p(x)\frac{dy}{dx}+q(x)y=f(x,y), \quad n=1,2 \quad y(0)=\alpha, \quad \frac{dy}{dx}=\beta \tag{1.1}$$

is considered in this work. The variables $p(x)$ and $q(x)$ are known smooth functions in a defined space while $f(x,y)$ is a nonlinear function that needed to be decomposed. Several methods have been developed and applied to solving this type of problems, Differential Transformation Method(DTM)(Bervillier, 2012), Homotopy perturbation Method(HPM)(Davood et al, 2008) and (He, 1999,2000), Power series(Guzel and Bayaram, 2005) to mention but few. Decomposition method however have also been widely applied in solving problems of this nature because it allows any function to be represented with a series, of which the terms can be recursively determined (Wazwaz, 2011).

Mahgoub Transform

Mahgoub Transform, though very recent has equally and widely been applied to analytically solve some problems in different types of equations such as linear and nonlinear forms of differential and integro-differential equations (Senthil and Viswanathan, 2016).

Mahgoub Integral transform is derived from Fourier integral with time domain with intention to improve and fasten the processes and procedures of finding solutions to these equations. This transform has its own properties (Mahgoub, 2016). Considering a set of functions with time domain and for any set of piecewise and exponential order functions, Mahgoub Transform is given as

$$A = (f(t) : \exists M, k_1, k_2 > 0. \mid f(t) \mid < Me^{\frac{|t|}{k_j}} \quad if \quad t \in (-1)^j x[0, \infty)) \tag{1.2}$$

with the constant $M \geq 0$, while k_1, and k_2 may be finite or infinite.

Let $f \in A$ then the Mahgoub transform is defined as

$$M[f(t)] = H(v) = v\int_0^{\infty} f(t)e^{-vt}dt \qquad t \geq 0, \qquad k_1 \leq v \leq k_2 \tag{1.3}$$

Mahgoub Transform Properties

Mahgoub Transform of Some functions

Table 1: Mahgoub Transform of some functions

$f(t)$	1	t	t^2	$t^n, \quad n \geq$	e^{at}	$Cosat$	
$M[f(t)]$	1	$\frac{1}{v}$	$\frac{2!}{v^2}$	$\frac{n!}{v^n}$	$\frac{v}{v-a}$	$\frac{v^2}{v^2+a^2}$	$\frac{a}{v^2 \cdot}$

Mahgoub Transform of Derivative Properties

Let $M[f(t)]$ be the Mahgoub Transform. Then

- $M[f'(t)] = vH(v) - vf(0)$
- $M[f''(t)] = v^2H(v) - vf'(0) - v^2 f(0)$
- $M[f^{(n)}(t)] = v^{(n)}H(v) - \sum_{k=0}^{n-1} v^{(n-k)} f^{(k)}(0)$

Convolution Theorem for Maghoub Transforms

Given two functions $F(t)$ and $G(t)$, the convolution of the two functions is defined as

$$F(t) * G(t) = \int_0^t F(x)G(t-x)dx = \int_0^t F(t-x)G(x)dx \tag{1.4}$$

For convolution theorem in Maghoub Transforms, let $M[F(t)] = H(v)$ and $N[G(t)] = I(v)$ then

$$M[F(t)*G(t)] = \frac{1}{v}M[F(t)]m[G(t)] = \frac{1}{v}H(v)I(v) \tag{1.5}$$

Pade Approximations

It is observed that Polynomials tend to be oscillatory, which causes errors. An attempts may be made to fix these errors but not in all cases. This, then becomes a disadvantage to polynomials approximations. This is what motivate us to use the rational function approximation that is more richer than polynomial approximations. A Pade rational approximation to $f(x)$ on the closed interval $[a,b]$ is defined as the quotient of two polynomials $P_n(x)$ and $Q_m(x)$ of degrees n and m respectively such that

$$f(x) \approx R_{[n,m]} = \frac{P_n(x)}{Q_m(x)} = \frac{\sum_{i=0}^{n} P_i x^i}{1 + \sum_{j=0}^{m} Q_i x^i} \tag{1.6}$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m, respectively. Letting $N = n + m$, the expectation is that any rational approximation of degree N will perform result-wise better or at least as good as any polynomial approximation of same degree.

METHODOLOGY

Considering (1.1), we transform it by (1.3) using operator denoted $M(.)$ to obtain

$$M[y''] + M[p(x)y'] + M[q(x)y] = M[f(x,y)] \tag{2.1}$$

We apply property 2 as defined in Section 1.2.2 to obtain

$$v^2M(y) - vy'(0) - v^2y(0) + M[p(x)y'] + M[q(x)y] = M[f(x,y)] \tag{2.2}$$

Now, substituting the initial conditions from (1.1), we have

$$v^2M(y) - v\beta - v^2\alpha + M[p(x)y'] + M[q(x)y] = M[f(x,y)] \tag{2.3}$$

and

$$M[y] = \alpha + \frac{\beta}{v} - \frac{1}{v^2}M[p(x)y'] - \frac{1}{v^2}M[q(x)y] + \frac{1}{v^2}M[f(x,y)] \tag{2.4}$$

We now introduce Mahgoub Transform Decomposition Method(MTDM) that allows the solution to (2.4) be represented by an infinite series and the nonlinear function be decomposed in form of Adomian polynomials.

We represent the approximate solution as

$$y = \sum_{n=0}^{\infty} y_n \tag{2.5}$$

The terms $y_n.s$ will be computed recursively.

We again represent the nonlinear operator $f(y)$ by

$$f(y) = \sum_{n=0}^{\infty} A_n \tag{2.6}$$

where $A_n{}'s$ are Adomian polynomials and they depend on $\{y_i\}_{i=0}^{n}$.

The values $A_n{}'s$ are defined as in (Jun-Sheng Duan, 2010) by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f(\sum_{i=0}^{\infty} y_n)\lambda^n \right]_{\lambda=0}, \qquad n = 0,1,2,... \tag{2.7}$$

where $y(\lambda) = \sum_{i=0}^{\infty}) y_n \lambda^n$ and $f(y(\lambda)) = \sum_{i=0}^{\infty} A_n \lambda^n$.

The first few of these polynomials are generated as follows

$$\begin{aligned} A_0 &= f(y_0), \\ A_1 &= y_1 f'(y_0), \\ A_2 &= y_2 f'(y_0) + \frac{1}{2!} y_1^2 f''(y_0), \\ A_3 &= y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{1}{3!} y_1^2 f'''(y_0), \\ &... \end{aligned} \tag{2.8}$$

The Adomian decomposition series is substituted for the solution $y(x)$ while the Adomian polynomials series (2.8) is also substituted for $f(y)$ in (2.4) to yield

$$M\left[\sum_{n=0}^{\infty} y_n\right] = \alpha + \frac{\beta}{v} - \frac{1}{v^2} M\left[p(x)\sum_{n=0}^{\infty} y_n'\right] - \frac{1}{v^2} M\left[q(x)\sum_{n=0}^{\infty} y_n\right] + \frac{1}{v^2} M\left[\sum_{n=0}^{\infty} A_n\right] \tag{2.9}$$

we then have that

$$\sum_{n=0}^{\infty} M[y_n] = \alpha + \frac{\beta}{v} - \frac{1}{v^2}\sum_{n=0}^{\infty} M[p(x)y_n'] - \frac{1}{v^2}\sum_{n=0}^{\infty} M[q(x)y_n] + \frac{1}{v^2}\sum_{n=0}^{\infty} M[A_n] \tag{2.10}$$

Generating few terms for $n = 0,1,2,3....$ we then compare both sides that resulted as

$$M[y_0] = \alpha + \frac{\beta}{v} \tag{2.11}$$

$$M[y_1] = -\frac{1}{v^2} M[p(x)y_0'] - \frac{1}{v^2} M[q(x)y_0] + \frac{1}{v^2} M[A_0] \tag{2.12}$$

$$M[y_2] = -\frac{1}{v^2} M[p(x)y_1'] - \frac{1}{v^2} M[q(x)y_1] + \frac{1}{v^2} M[A_1] \tag{2.13}$$

Generally, we have that

$$M[y_{n+1}] = -\frac{1}{v^2}M[p(x)y_n'] - \frac{1}{v^2}M[q(x)y_n] + \frac{1}{v^2}M[A_n] \tag{2.14}$$

Applying the Mahgoub inverse transform to (2.11), we obtain

$$y_0 = \alpha + \beta x \tag{2.15}$$

which can now be substituted into (2.12) to yield

$$M[y_1] = -\frac{\alpha+\beta}{v^2}M[p(x)+q(x)] - \frac{\beta}{v^2}M[xq(x)] + \frac{1}{v^2}M[f(y_0)] \tag{2.16}$$

The Mahgoub transforms of (2.16) is then evaluated to obtain value for y_1 as

$$y_1 = M^{-1}\left\{-\frac{\alpha+\beta}{v^2}M[p(x)+q(x)] - \frac{\beta}{v^2}M[xq(x)] + \frac{1}{v^2}M[f(y_0)]\right\} \tag{2.17}$$

The other terms $y_n, n = 2,3,...$ are subsequently obtained recursively which are then summed together to obtain the approximate solution (2.5).

Numerical Examples and Results

Numerical examples were considered using the Maghoub Transform Decomposition Method (MTDM) and the results compared with the existing solutions of Laplace method.

Numerical Examples

Problem 3.1 *We consider the nonlinear problem*

$$y' + y^2 = 1 \qquad y(0) = 3$$

with exact solution of

$$y = \frac{2}{1 - \frac{1}{2}e^{-2x}}$$

Source:-(Suheil,2001)

Using property 2, we take the Mahgoub transform of the differential equation to obtain

$$M[y'] + M[y^2] = M[1]$$

$$vM[y] - vy(0) + M[y^2] = 1 \tag{3.1}$$

Applying the initial conditions, we have that

$$M[y] = 3 + \frac{1}{v} - \frac{1}{v}M[y^2] \tag{3.2}$$

Now, by using the assumed infinite series, we adopt (2.9) to obtain

$$M\left[\sum_{n=0}^{\infty} y_n\right] = 3 + \frac{1}{v} - \frac{1}{v^2}M\left[\sum_{n=0}^{\infty} A_n\right] \tag{3.3}$$

From the problem, the nonlinear operator is

$f(y) = y^2$ and by the decomposition technique of (2.5), the operator is discomposed and few terms of the polynomials are generated from (2.8) to give

$$
\begin{aligned}
A_0 &= y_0^2,\\
A_1 &= 2y_0y_1,\\
A_2 &= 2y_0y_2 + y_1^2,\\
A_3 &= 2y_0y_3 + 2y_1y_2,\\
&\dots
\end{aligned}
\tag{3.4}
$$

Expanding (3.3) iteratively and compare both side of the equations, we have that

$$M[y_0] = 3 + \frac{1}{v} \tag{3.5}$$

$$M[y_1] = -\frac{1}{v}M[A_0] \tag{3.6}$$

Generally, we have that

$$M[y_{n+1}] = -\frac{1}{v}M[A_n] \quad n = 2,3,4,\dots \tag{3.7}$$

To obtain the first term y_0, we take the Mahgoub inverse of(3.5) which is

$$y_0 = x + 3, \tag{3.8}$$

that is substituted into (3.5) to obtain the value of A_0 which in turn is substituted into (3.6) to obtain $M[y_1]$.

The procedure is repeated to recursively generate the remaining values $y_i, i = 2,3,4,\dots$ and we have that

$$y_1 = -\frac{1}{3}x^3 - 3x^2 - 9x. \tag{3.9}$$

We again use the value of Maghoub transform of y_1 in (3.9) to obtain

$$M[y_2] = -\frac{1}{v}M[2y_0y_1] = -\frac{1}{v}M[2(x+3)(-\frac{1}{3}x^3 - 3x^2 - 9x)] \tag{3.10}$$

which implies that

$$M[y_2] = \frac{16}{v^5} + \frac{48}{v^4} + \frac{72}{v^3} + \frac{54}{v^2} \tag{3.11}$$

Hence, recursive computations yield

$$y_2 = \frac{2}{15}x^5 + 2x^4 + 12x^3 + 27x^2$$
$$y_3 = -\frac{17}{315}x^7 - \frac{17}{15}x^6 - \frac{51}{5}x^5 - 45x^4 - 81x^3$$
$$y_4 = \frac{2}{135}x^9 + \frac{2}{5}x^8 + \frac{27}{5}x^6 + 18x^5 + 162x^4 \tag{3.12}$$

Now, the approximate solution for the problem becomes

$$y = y_0 + y_1 + y_2 + y_3 + y_4 \ldots$$
$$y = 3 - 8x + 24x^2 + \frac{46}{3}x^3 + 146x^4 + \frac{209}{15}x^5 + \frac{461}{15}x^6 + \frac{242}{45}x^7 + \frac{2}{5}x^8 + \frac{2}{135}x^9 \tag{3.13}$$

Problem 3.2 *We again consider the second order nonlinear equation of the form*

$$y'' + (1-x)y' - y = 2y^3 \qquad y(0) = 1 \qquad y'(0) = 1$$

whose exact solution is

$$y = \frac{1}{1-x}$$

Source:-(Suheil,2001)

Using property 2, we take the Mahgoub transform of the differential equation to obtain

$$M[y''] = M[(x-1)y'] + M[y] + 2M[y^3]$$
$$v^2M[y] - vy'(0) - v^2y(0) = M[(x-1)y'] + M[y] + 2M[y^3] \tag{3.14}$$

Applying the initial conditions, we have that

$$M[y] = \frac{1}{v} + 1 + \frac{1}{v^2}M[(x-1)y'] + M[y] + \frac{2}{v^2}M[y^3] \tag{3.15}$$

Now, by using the assumed infinite series, we adopt (2.7) to obtain

$$M\left[\sum_{n=0}^{\infty} y_n\right] = \frac{1}{v} + 1 + \frac{1}{v^2}M\left[(x-1)\sum_{i=0}^{\infty} y_n'\right] + M\left[\sum_{i=0}^{\infty} y_n\right] + \frac{2}{v^2}M\left[\sum_{n=0}^{\infty} A_n\right] \tag{3.16}$$

The nonlinear operator in this problem is $f(y) = y^3$ and is treated an as in Problem 3.1 while few of the Adomian polynomials generated are

$$A_0 = y_0^3,$$
$$A_1 = 3y_0^2y_1,$$
$$A_2 = 3y_0^2y_2 + 3y_0y_1^2,$$
$$A_3 = 3y_0^2y_3 + 6y_0y_1y_2 + y_1^3,$$
$$\ldots \tag{3.17}$$

Expansion of (3.16) is iteratively computed and both side of the equations were compared. We then have that

$$M[y_0]=1+\frac{2}{v} \tag{3.18}$$

$$M[y_1]=\frac{1}{v^2}M[(x-1)y_0']+\frac{1}{v^2}M[y_0]+\frac{1}{v^2}M[A_0] \tag{3.19}$$

$$M[y_2]=ac1v^2M[(x-1)y_1']+\frac{1}{v^2}M[y_1]+\frac{1}{v^2}M[A_1] \tag{3.20}$$

and generally,

$$M[y_{n+1}]=\frac{1}{v^2}M[(x-1)y_n']+\frac{1}{v^2}M[y_n]+\frac{1}{v^2}M[A_n]. \tag{3.21}$$

To obtain the first term y_0, we take the Mahgoub inverse of(3.18) that yields

$$y_0=1+x. \tag{3.22}$$

We therefore substitute the value of y_0 into (3.17) to obtain the value of A_0 which we in turn substituted into (3.19) to obtain

$$M[y_1]=\frac{1}{v^2}M[(x-1)]+\frac{1}{v^2}M[1+x]+\frac{2}{v^2}M[(1+x)^3]$$

$$M[y_1]=\frac{1}{v^2}M[2x]+\frac{2}{v^2}M[(1+x)^3] \tag{3.23}$$

The Mahgoub inverse transform of (3.23) now yields

$$y_1=x^2+\frac{4}{3}x^3+\frac{1}{2}x^4+\frac{1}{5}x^5. \tag{3.24}$$

We evaluate the value for y_2 using (3.24) and the polynomial expansion for A_1 in (3.17) to obtain

$$y_2=-\frac{1}{3}x^3+\frac{5}{12}x^4+\frac{7}{6}x^5+\frac{53}{60}x^6+\frac{1}{105}x^7+\frac{27}{280}x^8+\frac{1}{40}x^9. \tag{3.25}$$

Recursively, the scheme yield the value for y_3 as

$$M[y_3]=\frac{1}{v^2}M[x^2-5x^3+\frac{3}{4}x^4+\frac{337}{10}x^5+\frac{727}{12}x^6+\frac{1541}{35}x^7+\frac{3043}{140}x^8+\frac{263}{35}x^9 +\frac{1623}{700}x^{10}+\frac{39}{100}x^{11}] \tag{3.26}$$

and the inverse transform of (3.26) is

$$y_3 = \frac{1}{16}x^4 - \frac{1}{4}x^5 + \frac{1}{40}x^6 + \frac{337}{420}x^7 + \frac{727}{672}x^8 + \frac{1541}{2520}x^9 + \frac{3043}{12600}x^{10} + \frac{263}{3850}x^{11} + \frac{1623}{92400}x^{12} + \frac{1}{400}x^{13}. \quad (3.27)$$

Therefore, the approximate solution for the problem is given as

$$y = y_0 + y_1 + y_2 + y_3 + \ldots$$

$$y = 1 + x + x^2 + \frac{6}{13}x^3 + \frac{89}{24}x^4 + \frac{127}{120}x^5 + \frac{569}{240}x^6 + \frac{289}{210}x^7 + \frac{5059}{1680}x^8 + \frac{34}{555}x^9 + \frac{47951}{25200}x^{10} + \frac{3517}{5775}x^{11} + \frac{509}{11550}x^{12} + \frac{11}{15600}x^{13} \quad (3.28)$$

RESULTS AND DISCUSSION

The results arrived at by Laplace Transform method is recovered using Maghoub Transform method. This established a strong correlation between the two methods in finding both analytic and approximate numerical solutions to mathematical problems. The scheme (3.13) for Problem 3.1 correlates exactly as in (Suheil,2001) and Figure 2 confirms both the absolute and relative errors when compared with the exact solution.

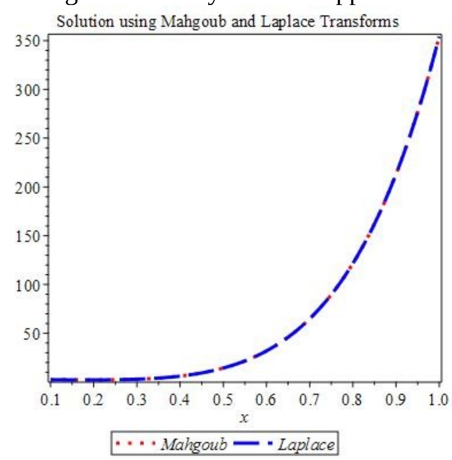

Fig. 1.Solution using the two Transforms

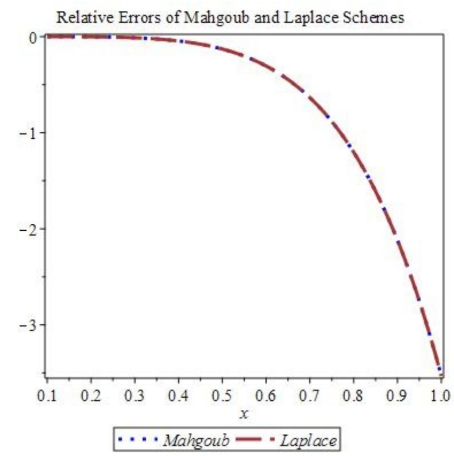

Fig. 2 Comparison of Errors with Exact Solution

As it was noted in (Suheil,2001), considering any value of $x > 1$, the solution $y(x)$ will diverge rapidly. An attempt is then made to convert the approximate solution to rational fraction approximation using Pade approximant of the form (1.3).

One of the advantages of Pade approximant as a form of polynomial approximation is that it allows approximation of any continuous function on a closed interval to within arbitrary tolerance. (Weierstrass approximation theorem)

We computed $[2,2]$ Pade approximant to the solution (3.13) to obtain

$$P_{[2,2]} = \frac{x^2 + x + 3}{\frac{1}{3}x^2 + 3x + 1} \quad (3.29)$$

and it also agreed with (Suheil,2001).

For Problem 3.2, Mahgoub scheme performed very well compared to Laplace algorithm. Fig 3 compared the the exact solution with Maghoub Scheme showing that it approximated the solution better than Laplace transform scheme as noted in (Suheil,2001)

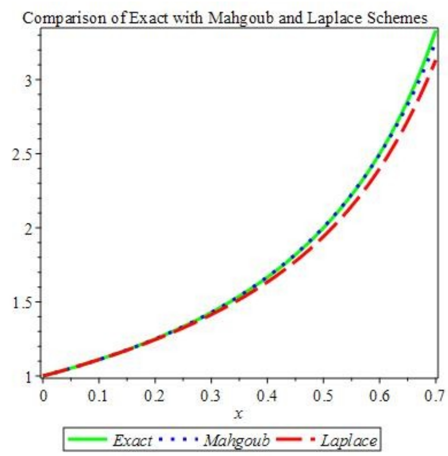

Fig. 3 Solution using the two Transforms

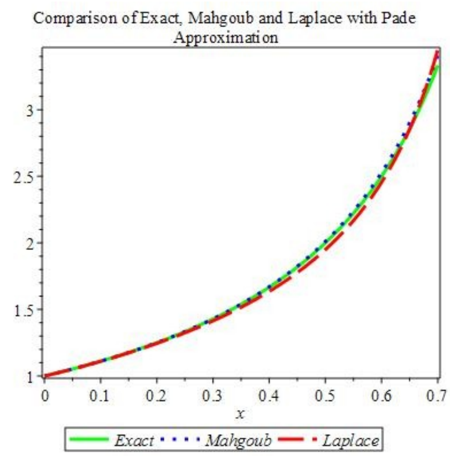

Fig. 4 $P_{[3,3]}$ Pade Approximant for the Schemes

However, due to rapid divergence noted in Problem 3.1, $P_{[3,3]}$ Pade Approximant is computed for Problem 3.2 which improved the rate of convergence for Mahgoub better than Laplace. We have that

$$P_{[3,3]} = \frac{1 + \frac{3}{5}x + \frac{163}{50}x^2 - \frac{1}{12}x^3}{1 - \frac{2}{5}x + \frac{133}{50}x^2 - \frac{1003}{300}x^3} \quad (3.30)$$

which converges faster than $P_{[5,5]}$ of Laplace method in (Suheil,2001).

CONCLUSION

Some examples of first order and second order of nonlinear differential equations are presented here to show the workability of the scheme. The results show that Mahgoub transform Decomposition Method is efficient and require relative less number of iterations before obtaining a better result that approximate the exact solution. Observations show that a better accuracy could be obtained if the number of the iterations is increased with application of Pade approximation technique in case the approximate solution will manifest oscillatory behavior.

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